Laplace Transformation

Let F(t)) be a real or complex valued function defined on $t \geq 0$. Then the function f(s) defined by $f(s) = \int_0^\infty e^{-st} F(t)dt$
is called Laplace Transformation of F(t) if the integral exists and $f(s) = L\{F(t)\}$.

Note: If $L\{F(t)\} = f(s)$ then F(t) is called the inverse Laplace Transform of f(s)
And we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the Laplace Transforms Operator.

What are the real worlds applications of Laplace transform?

a) Data mining/machine learning
b) System Modeling
c) Analysis of Electrical Circuits
d) Analysis of Electronic Circuits
e) Digital Signal Processing
f) Nuclear Physics
g) Process Controls

Prove that, $\mathbf{L\{e^{at}\} = \frac{1}{s-a}}$ **Solution:** By definition $L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$ So we have $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$ $= \int_0^\infty \mathbf{e^{-st+at}}\, dt$	$= \int_0^\infty \mathbf{e^{(-s+a)t}}\, dt$ $= \left[\frac{\mathbf{e^{(-s+a)t}}}{(-s+a)}\right]_0^\infty$ $= \left[0 - \left(\frac{\mathbf{1}}{(-s+a)}\right)\right]$ $= \frac{1}{s-a}$

Prove that, $\mathbf{L\{t\} = \frac{1}{s^2}}$ **Solution:** By definition $L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$ So we have $L\{e^{at}\} = \int_0^\infty e^{-st}\, t\, dt$ Now we have, $\int t\, e^{-st}\, dt$ $= t \int e^{-st}dt - \int \left\{\frac{d}{dt}(t) \int e^{-st}\, dt\right\} dt$ $= -\frac{t}{s}\, e^{-st} — (-\frac{1}{s} \int e^{-st}\, dt)$ $= -\frac{t}{s}\, e^{-st} + \frac{1}{s}\int e^{-st}dt$	$= -\frac{t}{s}\, e^{-st} + \frac{1}{s}\left(-\frac{1}{s}\, e^{-st}\right)$ $= -\frac{t}{s}\, e^{-st} - \frac{1}{s^2}\, e^{-st}$ $= e^{-st}\left(-\frac{t}{s} - \frac{1}{s^2}\right)$ **So we get** $\left[e^{-st}\left(-\frac{t}{s} - \frac{1}{s^2}\right)\right]_0^\infty$ $= \left[0 - \left(-\frac{0}{s} - \frac{1}{s^2}\right)\right]$ $= \frac{1}{s^2}$

Prove that, $\mathbf{L\{t^2\} = \frac{2}{s^3}}$	Formula $\mathbf{L\{t^n\} = \frac{n!}{s^{n+1}}}$

Prove that, $\mathbf{L\{\sin at\} = \frac{a}{s^2+a^2}}$

Solution:

By definition $L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$

So we have

$L\{\mathbf{\sin at}\} = \int_0^\infty e^{-st} \mathbf{\sin at}\, dt$

$= \left[\frac{e^{-st}(-s\sin at - a\cos at)}{\mathbf{s^2+a^2}}\right]_0^\infty$ [using the formula $\int e^{ax}\sin bx\ dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$]

$= \left[0 - \left(\frac{\mathbf{0-a}}{\mathbf{s^2+a^2}}\right)\right] = \frac{\mathbf{a}}{\mathbf{s^2+a^2}}$

Same problem:

Prove that, $\mathbf{L\{\cos at\} = \frac{s}{s^2+a^2}}$ [using the formula $\int e^{ax}\cos bx\ dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$]

Exercises

a) Prove that, $L\{3t - 5\} = \frac{s}{s^2} - 5s$

b) Prove that, $L\{2t^3 - 6t + 8\} = \frac{12}{s^4} - \frac{6}{s^2} + \frac{8}{s}$

c) Prove that, $L\{6\sin 2t - 5\cos 2t\} = \frac{12}{s^2+4} - \frac{6}{s^2+4} = \frac{12-5s}{s^2+4}$

d) Prove that, $L\{e^{-2t} - e^{-3t}\} = \frac{1}{s+2} + \frac{1}{s+3}$